Chapter 1

Matrices and Systems of Equations

1.1 System of Linear Equations

Pre-Questions:

Q1: How many unknowns and how many equations?

Can you solve the system of linear equations by your high school study?

$$\begin{cases} 2x_1 + x_2 = 7, \\ x_1 - 3x_2 = -2, \end{cases}$$

Q2: How many unknowns and how many equations?

Can you solve the system of linear equations by your high school study?

$$(1)\begin{cases} 3x_1 + 2x_2 - x_3 = -2, \\ x_2 = 3, \\ 2x_3 = 4. \end{cases}$$

$$(2)\begin{cases} 3x_1 + 2x_2 - x_3 = -2, \\ -3x_1 - x_2 + x_3 = 5, \\ 3x_1 + 2x_2 + x_3 = 2. \end{cases}$$

We give two new definitions:

A *linear equation in n unknowns* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers and $x_1, x_2, \ldots, x_n$ variables.

A *linear system of* m equations in n unknowns is then a system of the form

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m.
\end{cases} \quad (1)
$$

certainly, we can call (1) as $m \times n$ system.

For example, Q1 in pre-question is 2×2 system.

Next to go back Q2 in pre-questions,

We calculate (1) and (2) and find out they have same solution set $\{(-2, 3, 2)\}$

Definition 1.1. Two systems of equations involving the same variables are said to be equivalent if they have the same solution set.

Q1: We pick up the Q2(1) to do the following actions:

(a) Please interchange the first equation and the second equation, do we change the solution set?

(b) Both sides of the first equation we multiply by nonzero number 2, do we change the solution set?

(c) Please do the action $e_1+e_2\times 2$, then compose the new system, do we change the solution set?

So we have this three conclusions:

(1) The order in which any two equations are written may be interchanged.

(2) Both sides of an equation may be multiplied by the same nonzero real number.

(3) A multiple of one equation may be added to (or subtracted from) another.

Q2: Please observe this two systems:

$$(1)\begin{cases} 3x_1 + 2x_2 + x_3 = 1, \\ \qquad\quad x_2 - x_3 = 2, \\ \qquad\qquad\quad 2x_3 = 4. \end{cases}$$

$$(2)\begin{cases} 2x_1 - x_2 + 3x_3 - 2x_4 = 1, \\ \qquad x_2 - 2x_3 + 3x_4 = 2, \\ \qquad\qquad 4x_3 + 3x_4 = 3, \\ \qquad\qquad\qquad 4x_4 = 4 \end{cases}$$

What kind of shape for these two systems?

Definition 1.2. A system is said to be in **strict**

triangular form if in the kth equation the coefficients of the first $k-1$ variables are all zero and the coefficient of x_k is nonzero ($k = 1, \ldots, n$).

Q3: In Q2(2),

$$\begin{cases} 3x_1 + 2x_2 - x_3 = -2, \\ -3x_1 - x_2 + x_3 = 5, \\ 3x_1 + 2x_2 + x_3 = 2. \end{cases}$$

If we pick up the coefficients of the $x_i^\prime s$ to compose a (3×3) array

$$\begin{pmatrix} 3 & 2 & -1 \\ -3 & -1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

We refer this array as the *coefficient matrix* of the system.

The term *matrix* means simply a rectangular array of numbers. A matrix having m rows and n columns is said to be $m \times n$ matrix.

Definition 1.3. A rectangular array of $m \times n$ scalars a_{ij} $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ arranged in m rows and n columns and denotes by

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

is called an (m, n)-matrix, or simply a matrix. We refer to a_{ij} as an **element** (or an **entry**) in the i-th row and j-th column in A.

Table 1.1: Table showing amounts of four kinds of goods needed in three areas

	B_1	B_2	B_3	B_4
A_1	1	2	5	4
A_2	2	3	1	0
A_3	4	1	5	2

Matrix is very important in our life such as in economics and finance area. For example, we can compile the following table showing the amounts of four kinds of goods B_1, B_2, B_3 and B_4 which are to be shipped to three areas A_1, A_2 and A_3.

For simplicity we can write it as a (3×4)-matrix,

$$\begin{pmatrix} 1 & 2 & 5 & 4 \\ 2 & 3 & 1 & 0 \\ 4 & 1 & 5 & 2 \end{pmatrix}$$

We still go back to look Q3, and we will have the new matrix

$$\begin{pmatrix} 3 & 2 & -1 & \vdots & -2 \\ -3 & -1 & 1 & \vdots & 5 \\ 3 & 2 & 1 & \vdots & 2 \end{pmatrix}$$

Next we will study what kind of matrix of this?

We will refer to this new matrix as the *augmented matrix*.

In general, when an $m \times r$ matrix B is attached to an $m \times n$ matrix A in this way, the augmented matrix is denoted by $(A|B)$. Thus if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mr} \end{pmatrix}$$

then

$$(A|B) = \begin{pmatrix} a_{11} & \cdots & a_{1n} & \vdots & b_{11} & \cdots & b_{1r} \\ \vdots & & & \vdots & \vdots & & \\ a_{m1} & \cdots & a_{mn} & \vdots & b_{m1} & \cdots & b_{mr} \end{pmatrix}$$

With each system of equations we may associate an augmented matrix of the form

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} & \vdots & b_1 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & \vdots & b_m \end{pmatrix}$$

Elementary Row Operations

Q1: What's kind of action we did to A's rows in the following three matrices

(1)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(3)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

Definition 1.4. Let A a matrix of order n, then we call

(1) interchange any two rows of A

(2) kR_i where $k \neq 0$

(3) $R_i + lR_j$

as **elementary row operations**.

For example,

$$\begin{pmatrix} 3 & 2 & -1 & \vdots & -2 \\ -3 & -1 & 1 & \vdots & 5 \\ 3 & 2 & 1 & \vdots & 2 \end{pmatrix}$$

Qa:Please finish these several actions $R_2 + R_1$, $R_3 - R1$ and what do you find?

So we give the first row as *pivotal row* because we find that the first row is used to eliminate the elements in the first column of the remaining rows. Certainly, the first nonzero entry in the pivotal row is called the *pivot*.

Qb:How did you find the solution set of pre-question Q2(2)? Please compare with Qa, what did you find in these two question?

Exercise: Can you find the new technique for solving this question?

Solve the system

$$\begin{cases} -x_2 - x_3 + x_4 = 0, \\ x_1 + x_2 + x_3 + x_4 = 6, \\ 2x_1 + 4x_2 + x_3 - 2x_4 = -1, \\ 3x_1 + x_2 - 2x_3 + 2x_4 = 3 \end{cases}$$

1.2 Row Echolon Form

Previously on Class:

New counting technique for solving system of linear equations:

$$\frac{\text{system of linear equations}}{\text{augmented matrix}} \to \frac{\text{strict triangular form}}{\text{new augmented matrix}}$$

$$\to \frac{\text{back substitution}}{\text{back substitution}}$$

For example,

Solve the following system:

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 0, \\ -2x_1 + x_2 - x_3 = 2, \\ 2x_1 - x_2 + 2x_3 = -1. \end{cases}$$

Augmented Matrix:

$$\begin{pmatrix} 3 & 2 & 1 & \vdots & 0 \\ -2 & 1 & -1 & \vdots & 2 \\ 2 & -1 & 2 & \vdots & -1 \end{pmatrix} \xrightarrow{R_1 + R_2} \begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{0} & \vdots & \mathbf{2} \\ -2 & 1 & -1 & \vdots & 2 \\ 2 & -1 & 2 & \vdots & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 3 & 0 & \vdots & 2 \\ \mathbf{0} & \mathbf{7} & \mathbf{-1} & \vdots & \mathbf{6} \\ 0 & -7 & 2 & \vdots & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 & \vdots & 2 \\ 0 & 7 & -1 & \vdots & 6 \\ 0 & 0 & 1 & \vdots & 1 \end{pmatrix}$$

$$\to \begin{cases} x_1 + 3x_2 \qquad = 2, \\ \quad 7x_2 - x_3 = 6, \\ \qquad\qquad x_3 = 1 \end{cases}$$

So $(-1, 1, 1)$

Q1: Consider the system represented by the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \vdots & 1 \\ -1 & -1 & 0 & 0 & 1 & \vdots & -1 \\ -2 & -2 & 0 & 0 & 3 & \vdots & 1 \\ 0 & 0 & 1 & 1 & 3 & \vdots & 3 \\ 1 & 1 & 2 & 2 & 4 & \vdots & 4 \end{pmatrix}$$

Please find **pivotal row** and **pivot** firstly, then reduce it one step by one step. what did you find in this matrix reduction?

At last we get

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \vdots & 1 \\ 0 & 0 & 1 & 1 & 2 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 3 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{pmatrix} \to \begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 1, \\ x_3 + x_4 + 2x_5 = 0, \\ x_5 = 3. \end{cases}$$

Q2: If we say (4,0,-6,0,3) is a solution of the system. Does it has only one solution or not?

Give two **definitions:**

Lead variables: The variables corresponding to the first nonzero elements in each row of the augmented matrix will be referred to as *lead variables*. So x_1, x_3, x_5 are lead variables.

The remaining variables corresponding to the columns skipped in the reduction process will b referred to as *free variables*. For example, x_2, x_4.

Row Echelon Matrix

Definition 1.5. (i) If the first nonzero entry in each nonzero row is 1.

(ii) If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.

(iii) If there are rows whose entries are all zero, they are below the rows having nonzero entries.

Example 1.6. Can you pick up all row echelon form in the following matrices:

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Definition 1.7. The process of using row operations (1), (2) and (3) to transform a linear system into one whose augmented matrix is in row echelon form is called **Gaussian elimination**.

Exercise1:

Please use the new technique (use elementary row operation to reduce the augmented matrix to row echelon matrix)you studied to calculate the following several system.

$$(1)\begin{cases} x_1 + x_2 = 1, \\ x_1 - x_2 = 3, \\ -x_1 + 2x_2 = -2. \end{cases} \quad (2)\begin{cases} x_1 + 2x_2 + x_3 = 1, \\ 2x_1 - x_2 + x_3 = 2, \\ 4x_1 + 3x_2 + 3x_3 = 4, \\ 2x_1 - x_2 + 3x_3 = 5. \end{cases}$$

$$(3)\begin{cases} x_1 + 2x_2 + x_3 = 1, \\ 2x_1 - x_2 + x_3 = 2, \\ 4x_1 + 3x_2 + 3x_3 = 4, \\ 3x_1 + x_2 + 2x_3 = 3 \end{cases}$$

Be careful: how many unknowns and equations in each question? How many solutions for each question?

A linear system is said to be *overdetermined* if there are more equations than unknowns.

Clearly, (1) is *inconsistent* (no solution) and (2),(3) are *consistent* (has at least one solution).

Exercise2:

(3) $$\begin{cases} x_1 + 2x_2 + x_3 = 1, \\ 2x_1 + 4x_2 + 2x_3 = 3. \end{cases}$$

(4) $$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 2, \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3, \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 2 \end{cases}$$

A system of m linear equations in n unknowns is said to be *underdetermined* if there are fewer equations than unknowns($m < n$).

Reduced Row Echelon Form

Example 1.8. Please solve the system

$$\begin{cases} -x_1 + x_2 - x_3 + 3x_4 = 0, \\ 3x_1 + x_2 - x_3 - x_4 = 0, \\ 2x_1 - x_2 - 2x_3 - x_4 = 0 \end{cases}$$

Solution:

$$\begin{pmatrix} \textbf{-1} & \textbf{1} & \textbf{-1} & \textbf{3} & \vdots & \textbf{0} \\ 3 & 1 & -1 & -1 & \vdots & 0 \\ 2 & -1 & -2 & -1 & \vdots & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & -1 & 3 & \vdots & 0 \\ \textbf{0} & \textbf{4} & \textbf{-4} & \textbf{8} & \vdots & \textbf{0} \\ 0 & 1 & -4 & 5 & \vdots & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 1 & -1 & 3 & \vdots & 0 \\ 0 & 1 & -1 & 2 & \vdots & 0 \\ 0 & 0 & -3 & 3 & \vdots & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & -3 & \vdots & 0 \\ 0 & 1 & -1 & 2 & \vdots & 0 \\ 0 & 0 & 1 & -1 & \vdots & 0 \end{pmatrix}$$

[row echelon form]

Give the continued reduction:

$$\to \begin{pmatrix} 1 & -1 & 1 & -3 & \vdots & 0 \\ 0 & 1 & -1 & 2 & \vdots & 0 \\ 0 & 0 & 1 & -1 & \vdots & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 1 & -1 & \vdots & 0 \end{pmatrix}$$

It is easy to see: suppose $x_4 = \alpha$, then $x_1 = \alpha, x_2 = -\alpha, x_3 = \alpha$.

So we have the solutions $(\alpha, -\alpha, \alpha, \alpha)$

Definition 1.9. A matrix is said to be in **reduced row echelon form** if:

(i) The matrix is in row echelon form.

(ii) The first nonzero entry in each row is the only nonzero entry in its column.

The following matrices are in reduced row echelon form:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The process of using elementary row operations to transform a matrix into reduced row echelon form is called *Gauss-Jordan reduction.*

Exercise3:

Use Gauss-Jordan reduction to solve the system:

$$\begin{cases} x_1 + 3x_2 + x_3 + x_4 = 3, \\ 2x_1 - 2x_2 + x_3 + 2x_4 = 8, \\ 3x_1 + x_2 + 2x_3 - x_4 = -1 \end{cases}$$

1.3 Matrix Algebra

Matrix Notation

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

A is an $m \times n$ matrix. We will sometimes shorten this to $A = (a_{ij})$. Simliarly, a matrix B may be referred to as (b_{ij}), a matrix C as (c_{ij}), and so on.

Vectors

2-tuple vector (2,3)

3-tuple vector (3,1,3)

4-tuple vector (2,1,3,4)

$\vdots$

n-tuple vector $(1, 2, 3, \ldots, n)$

A solution to a system of m linear equations in n unknowns is an n-tuple of real numbers. we will refer to an n-tuple of real numbers as a vector. If n-tuple is represented in terms of a $1 \times n$ matrix, then we will refer to it as a *row vector*. Alternatively, if the n-tuple is represented by an $n \times 1$ matrix, then we refer it to as a column vector.

For example, the solution of

$$\begin{cases} x_1 + x_2 = 3, \\ x_1 - x_2 = 1 \end{cases}$$

can be shown in (2,1) or $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Equality

Definition 1.10. Two $m \times n$ matrices A and B are said to be **equal** if $a_{ij} = b_{ij}$ for each i and j.

Scalar Multiplication

Definition 1.11. If A is a matrix and α is a scalar, then αA is the $m \times n$ matrix whose (i, j) entry is αa_{ij}.

For example, if

$$A = \begin{pmatrix} 4 & 8 & 2 \\ 6 & 8 & 10 \end{pmatrix}$$

then

$\frac{1}{2}A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 4 & 5 \end{pmatrix}$ and $3A = \begin{pmatrix} 12 & 24 & 6 \\ 18 & 24 & 30 \end{pmatrix}$

Matrix Addition

Definition 1.12. If $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ matrices, then the sum $A + B$ is the $m \times n$ matrix whose i, j entry is $a_{ij} + b_{ij}$ for each ordered pair (i, j).

For example,

$$A + B = \begin{pmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{pmatrix} + \begin{pmatrix} -2 & 1 & 3 \\ -4 & -3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 5 \\ -8 & 0 & 2 \\ 3 & 4 & 8 \end{pmatrix}$$

For example,

$$A - B = \begin{pmatrix} 4 & 0 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

Zero Matrix

The matrix whose elements are all zero is the **zero matrix**, denoted by O. Then we have

(1) $A + O = A$

(2) $A - A = A + (-A) = O$

For example,

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

⋆Matrix Multiplication and Linear Systems

Case 1 One equation in Several Unknowns

For example,

$$3x_1 + 2x_2 + 5x_3 = 4$$

Let's re-compose the equation.

Suppose $A = (3, 2, 5)$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Define the **product** by

$$A\mathbf{x} = (3, 2, 5) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3x_1 + 2x_2 + 5x_3.$$

Then we rewrite the equation

$$A\mathbf{x} = 4$$

Q1: If give you a linear equation with n unknowns of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Please re-compose the equation.

Solution:

For example, if

$$A = (2, 1, -3, 4), \text{ and } \mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ -2 \end{pmatrix}$$

Please substitute A and $\mathbf{x}$ in the definition of product. What's the last result?

Case 2: M equations in N Unknowns

Consider

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

If we do the similar re-composition to this system, that is, we want it as $A\mathbf{x} = b$.

Can you give the proper A, $\mathbf{x}$ and $\mathbf{b}$ here?

Please finish these several examples:

Example 1.13.

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 5 & 3 & 7 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Give $A\mathbf{x} =$ __________

Example 1.14.

$$A = \begin{pmatrix} -3 & 1 \\ 2 & 5 \\ 4 & 2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Give $A\mathbf{x} =$ _________

Example 1.15. Write the following system of equations as a matrix equation $A\mathbf{x} = b$.

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 5, \\ x_1 - 2x_2 + 5x_3 = -2, \\ 2x_1 + x_2 - 3x_3 = 1 \end{cases}$$

New way:

$$A\mathbf{x} = x_1 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$$

Definition 1.16. If $\mathrm{a}_1, \ldots, \mathrm{a}_n$ are vectors in R^m and $c_1, \ldots, c_n$ are scalars, then a sum of the form

$$c_1\mathrm{a}_1+c_2\mathrm{a}_2+\cdots+c_n\mathrm{a}_n$$

is said to be a **linear combination** of the vectors $\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_n$.

Example 1.17. The linear system

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 = 5, \\ 5x_1 - 4x_2 + 2x_3 = 6 \end{cases}$$

Certainly, we can write it into

$$x_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -4 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

If we choose $x_1 = 2, x_2 = 3, x_3 = 4$, then we go on to get

$$\begin{pmatrix} 5 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ -4 \end{pmatrix} + 4 \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

Thus the vector $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ is a *linear combination* of the three column vectors of the coefficient matrix. Certainly, we can say the linear system is consistent and

$$\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

is a solution to the system.

Theorem 1.18. ***(Consistency Theorem for Linear Systems)*** *A linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ can be written as a linear combination of the column vectors of A.*

Example 1.19. We know

$$\begin{cases} x_1 + 2x_2 = 1, \\ 2x_1 + 4x_2 = 1 \end{cases}$$

has no solution, so we say it is inconsistent. Because we cannot write the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as linear combination of the column vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Product of two matrices

Firstly, please observe these two matrices

$$A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 \\ 1 & -2 \\ 3 & 1 \end{pmatrix},$$

Q1:

(1) size of A? size of B?

(2) Can we do AB? How to do it?

Firstly, check these calculations:

$$a_{11} = \begin{pmatrix} 1 & -2 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$
$$= 1 \times 2 + (-2) \times 1 + (-3) \times 3 = -9$$

$$a_{12} = \begin{pmatrix} 1 & -2 & -3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$= 1 \times 3 + (-2) \times (-2) + (-3) \times 1 = 4$$

$$a_{21} = \begin{pmatrix} 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$= 2 \times 2 + (-1) \times 1 + 0 \times 3 = 3$$

$$a_{22} = \begin{pmatrix} 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$= 2 \times 3 + (-1) \times (-2) + 0 \times 1 = 8$$

Then we give **product** of AB, i.e.

AB

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 2 + (-2) \times 1 + (-3) \times 3 & 1 \times 3 + (-2) \times (-2) + (-3) \times 1 \\ 2 \times 2 + (-1) \times 1 + 0 \times 3 & 2 \times 3 + (-1) \times (-2) + 0 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} -9 & 4 \\ 3 & 8 \end{pmatrix}$$

(3) Size of AB? Can you find the relation among size of A, B and AB? If you cannot, please do the following question carefully.

Answer 1:

size of A (2,3) $\cdots$ (3,2) size of B

size of AB: (2,2)

If

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 & 4 \\ 3 & -2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Q2. Please calculate AB? And what's the size of AB? Can you find the relation among size of A, B and AB so far?

Answer 2:

size of A (2,3) $\cdots$ (3,3) size of B

size of AB: (2,3)

Q3. Is there solution for BA?

We will continue to use this calculation in next question.

So we can give the definition of **product** of matrices

Definition 1.20. The **product** of two matrices

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{ml} \end{pmatrix}, B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{l1} & \cdots & b_{ln} \end{pmatrix}$$

is

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \cdots & \cdots & \cdots \\ c_{m1} & \cdots & c_{mn} \end{pmatrix},$$

where

$$c_{ij} = a_{i1}b_{1j} + \cdots + a_{il}b_{lj} = \sum_{t=1}^{l} a_{it}b_{tj}$$

The size of AB:

$$(m,l)\cdot(l,n)=(m,n)$$

Q4: Can you calculate BA in **Q2**? and $AB = BA$?

Q5: If

$$A=\begin{pmatrix}1&0&3\\2&1&-1\end{pmatrix}, B=\begin{pmatrix}-1&1&4\\3&-2&1\\0&0&2\end{pmatrix}, C=\begin{pmatrix}2\\1\\0\end{pmatrix}$$

then to Check
a. $A(BC)$ and $(AB)C$

If

$$A=\begin{pmatrix}1&0&3\\2&1&-1\\1&1&2\end{pmatrix}, B=\begin{pmatrix}-1&1&4\\3&-2&1\\0&0&2\end{pmatrix}, C=\begin{pmatrix}2\\1\\0\end{pmatrix}$$

b. $(A+B)C$ and $AC+BC$

Algebraic Rules

So we will get the following **properties** of matrix

(1) $AB \neq BA$

(2) $A(BC) =(AB)C$

(3) $(A + B)C = AC + BC$

(4) $A(B + C) = AB + AC$

And the other properties:

(5) $A + B = B + A$

(6) $(A + B) + C =A + (B + C)$

(7) $(\alpha\beta)A = \alpha(\beta A)$

(8) $\alpha(AB) = (\alpha A)B = A(\alpha B)$

At the same time we can have this conclusion:

If A is an (m, n)-matrix and B is an (n, p)-matrix, then AB is an (m, p)-matrix. But BA may not be defined when $p \neq m$.

Example 1.21. If

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Give $A^2 =$ __________

Give $A^3 =$ __________

$\vdots$

Give $A^n =$ __________

Conclusion: $A^k = \underbrace{AA \cdots A}_{k \text{ times}}$

The Identity Matrix

Identity matrix plays the same role to 1 in real number system.

Please check these several questions:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

AI and IA?

Definition 1.22. The $n \times n$ **identity matrix** is the matrix $I = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

In general, If B is $m \times n$ matrix and C is an $n \times r$ matrix, then

$$BI = B \text{ and } IC = C$$

For example,

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$$

We usually write $n \times n$ identity matrix I are the standard vectors e_j which is the jth column vector of I.

$$I = (e_1, e_2, \cdots, e_n)$$

Matrix Inversion

Definition 1.23. An $n \times n$ matrix A is said to be **nonsingular** or **invertible** if there exists a matrix B such that $AB = BA = I$. The matrix B is said to be a **multiplicative inverse** of A.

If B and C are both multiplicative inverse of A, then

$$B = BI = B(AC) = (BA)C = IC = C$$

Thus a matrix can have at most one multiplicative inverse as simply the *inverse* of A and denoted it by A^{-1}.

Qa: Please give the result of

$$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

Qb: Does $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ have an inverse or not? Reason?

Theorem 1.24. *If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.*

Proof

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}B = I$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$$

Certainly, we have the following conclustion:

$$(A_1A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1}A_1^{-1}$$

The transpose of a Matrix

Observe these pairs of matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 3 & -1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Definition 1.25. The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix B defined by

$$b_{ji} = a_{ij}$$

for $j = 1, 2, \ldots, n$ and $i = 1, \ldots, m$. The transpose of A is denoted by A^T.

Algebraic Rules for Transposes

1. $(A^T)^T = A$
2. $(\alpha A)^T = \alpha(A)^T$
3. $(A + B)^T = A^T + B^T$
4. $(AB)^T = B^T A^T$

We use an example to check 4.

Example 1.26. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ \mathbf{2} & \mathbf{4} & \mathbf{1} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} & 2 \\ 2 & \mathbf{1} & 1 \\ 5 & \mathbf{4} & 1 \end{pmatrix} = \begin{pmatrix} 10 & 6 & 5 \\ 34 & 23 & 14 \\ 15 & \mathbf{8} & 9 \end{pmatrix}$$

When the product is transposed, the (3,2) entry of AB becomes the (2,3) entry of $(AB)^T$.

$$(AB)^T = \begin{pmatrix} 10 & 34 & 15 \\ 6 & 23 & \mathbf{8} \\ 5 & 14 & 9 \end{pmatrix}$$

On the other hand, the (2,3) entry of $B^T A^T$ is computed using the second row of B^T and the third column of A^T.

$$B^T A^T = \begin{pmatrix} 1 & 2 & 5 \\ \mathbf{0} & \mathbf{1} & \mathbf{4} \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & \mathbf{2} \\ 2 & 3 & \mathbf{4} \\ 1 & 5 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 10 & 34 & 15 \\ 6 & 23 & \mathbf{8} \\ 5 & 14 & 9 \end{pmatrix}$$

Qc: Please check these matrices:

if $A =$

$$\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & -2 \\ 2 & -2 & -3 \end{pmatrix}$$

then A^T respectively?

Definition 1.27. An $n \times n$ matrix A is said to be **symmetric** if $A^T = A$.

1.4 elementary matrix

Definition 1.28. If we do elementary operation to an identity matrix I once, then the new matrix we get is elementary matrix.

For example:

(1)

$$\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 0 & \dots & \dots & & 1 & & \\ & & \vdots & 1 & & & \vdots & & \\ & & \vdots & & \ddots & & \vdots & & \\ & & \vdots & & & 1 & \vdots & & \\ & & 1 & \dots & \dots & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & k & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

(3)

$$\begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & \dots & \dots & l & & \\ & & & \ddots & & \vdots & & \\ & & & & \ddots & \vdots & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

Property: Elementary matrices are invertible.

Observe these questions:

(1) Suppose

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \to A_1 = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

meanwhile

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix} = ?$$

$$\text{answer} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Conclusion: $A_1 = EA$ (premultiplying)

(2)

$$A = \begin{pmatrix} 2 & 6 & -4 \\ -3 & -6 & 5 \\ 2 & 2 & -2 \end{pmatrix} \to A_3 = \begin{pmatrix} -6 & 6 & -4 \\ 7 & -6 & 5 \\ -2 & 2 & -2 \end{pmatrix}$$

meanwhile

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 6 & -4 \\ -3 & -6 & 5 \\ 2 & 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = ?$$

$$\text{answer} = \begin{pmatrix} -6 & 6 & -4 \\ 7 & -6 & 5 \\ -2 & 2 & -2 \end{pmatrix}$$

Conclusion: $A_3 = AE$ (postmultiplying)

By two conclusions above we have

Theorem 1.29. *The matrix obtained by performing row(column) elementary operations on a matrix A is equal to the matrix obtained by premultiplying(postmultiplying) A by a corresponding elementary matrix.*

Definition 1.30. A matrix B is **row equivalent** to A if there exists a finite sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that

$B = E_k E_{k-1} \ldots E_1 A$

Two augmented matrices $(A|b)$ and $(B|c)$ are row equivalent if and only if $Ax = b$ and $Bx = c$ are equivalent systems.

Theorem 1.31. ***(Equivalent Conditions***

for Nonsingularity) *Let A be an $n \times n$ matrix. The following are equivalent:*

(a) A is nonsingular.

(b) $Ax = 0$(homogeneous system) has only the trivial solution $\boldsymbol{0}$.

(c) A is row equivalent to I.

Theorem 1.32. *The system of n linear equations in n unknowns $Ax = b$(nonhomogeneous) has a unique solution if and only if A is nonsingular.*

By the theorem 1.31 any nonsingular matrix A is equivalent to I, we have

$$E_m \cdots E_1 A = I \tag{1}$$

From the uniqueness of inverse matrix, we get $E_m \cdots E_1 = A^{-1}$, i.e.,

$$E_m \cdots E_1 I = A^{-1} \tag{2}$$

(1) gives us information which is to do elementary operations $E_m \cdots E_1$ to A,we can have A.

Meanwhile (2) shows us do elementary operations $E_m \cdots E_1$ to I, we can have A^{-1}.

NEXT to introduce a new technique for getting inverse matrix:

$$(A \vdots E) \overset{\text{elementary operations on rows}}{\longrightarrow} (E \vdots A^{-1})$$

Example 1.33. Suppose

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -3 & 2 & -5 \end{pmatrix}$$

what's A^{-1}?

Solution:

We have 3×6 block matrix

$$(A \vdots I) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -3 & 2 & -5 & 0 & 0 & 1 \end{pmatrix}$$

Next to do some elementary operations and we will get $(I \vdots A^{-1})$,

$$\xrightarrow{R_2+(-2)\times R_1} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ -3 & 2 & -5 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3+3\times R_1} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 2 & -2 & 3 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3+(-2)\times R_2} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 2 & 7 & -2 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & -1 & 1 \\ 0 & 0 & 2 & 7 & -2 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1+(-1/2)R_3} \begin{pmatrix} 1 & 0 & 0 & -\frac{5}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & 5 & -1 & 1 \\ 0 & 0 & 2 & 7 & -2 & 1 \end{pmatrix}$$

$$\xrightarrow{(1/2)R_3} \begin{pmatrix} 1 & 0 & 0 & -\frac{5}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & 5 & -1 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

then we have

$$A^{-1} = \begin{pmatrix} -\frac{5}{2} & 1 & -\frac{1}{2} \\ 5 & -1 & 1 \\ \frac{7}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

Example 1.34. Solve the system

$$\begin{cases} x_1 + x_3 = 3, \\ 2x_1 + x_2 = 8, \\ -3x_1 + 2x_2 - 5x_3 = -1 \end{cases}$$

Solution:

$$x = A^{-1}b = \begin{pmatrix} -\frac{5}{2} & 1 & -\frac{1}{2} \\ 5 & -1 & 1 \\ \frac{7}{2} & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}$$

Triangular Matrix

Like determinants, a matrix of order n in the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}$$

is called **an upper triangular matrix**. Certainly,

$$\begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is called **a lower triangular matrix**.

What are about kA, $A + B$ and AB if A, B are upper triangular matrices?

Diagonal Matrix

A matrix of order n whose elements are zeros except for elements on the main diagonal:

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$$

is called **a diagonal matrix**. When $a_1 = a_2 = \cdots = a_n$, it is a scalar matrix. It is easy to see from the definition that

$$k \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} = \begin{pmatrix} ka_1 & & & \\ & ka_2 & & \\ & & \ddots & \\ & & & ka_n \end{pmatrix}$$

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} + \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{pmatrix}$$

$$= \begin{pmatrix} a_1+b_1 & & & \\ & a_2+b_2 & & \\ & & \ddots & \\ & & & a_n+b_n \end{pmatrix}$$

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{pmatrix}$$

$$= \begin{pmatrix} a_1b_1 & & & \\ & a_2b_2 & & \\ & & \ddots & \\ & & & a_nb_n \end{pmatrix}$$

If A is a diagonal matrix, then $A^T = A$.

Triangular Factorization

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix}$$

Let step 1: $l_{21} = 1/2$, $l_{31} = 2$.

step 2: $l_{32} = -3$.

Suppose the result matrix $U = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix}$,

and $L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix}$

Q: Please check LU?

Conclusion: $E_3 E_2 E_1 A = U$

where

$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$,

$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$.

Certainly, we have $A = E_1^{-1} E_2^{-1} E_3^{-1} U$, please check

$E_1^{-1} E_2^{-1} E_3^{-1}$? What did you find?

Partitioned Matrices

Although some matrices are not diagonal, they can be partitioned into **block diagonal matrices**:

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{pmatrix}$$

where the orders of matrices $A_1, A_2, \cdots, A_m$ may be the same or different.

For example,

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 3 & 1 \\ 0 & 0 & 3 \end{array}\right) = \begin{pmatrix} 2 & & \\ & 3 & 1 \\ & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right) = \begin{pmatrix} 2 & 0 & & \\ 1 & 2 & & \\ & & 3 & 0 \\ & & 1 & 3 \end{pmatrix}$$

are block diagonal matrices.

If matrices A and B of order n are block diagonal matrices

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_m \end{pmatrix},$$

where A_i and B_i, $i = 1, 2, \ldots, m$, are matrices of the same order. It is not difficult to show by definition that

$$k \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{pmatrix} = \begin{pmatrix} kA_1 & & & \\ & kA_2 & & \\ & & \ddots & \\ & & & kA_m \end{pmatrix},$$

$$A+B = \begin{pmatrix} A_1 + B_1 & & & \\ & A_2 + B_2 & & \\ & & \ddots & \\ & & & A_m + B_m \end{pmatrix},$$

$$AB = \begin{pmatrix} A_1B_1 & & & \\ & A_2B_2 & & \\ & & \ddots & \\ & & & A_mB_m \end{pmatrix}.$$

So, the sum, scalar multiplication and product of block diagonal matrices can be computed just like diagonal matrices.

For example,

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2+1 & \\ & \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & & \\ & 5 & 1 \\ & 3 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}^2 = \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}^2 & \\ & \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}^2 \end{pmatrix}$$

$$
= \begin{pmatrix} \begin{pmatrix} 4 & 0 \\ 4 & 4 \end{pmatrix} & \\ & \begin{pmatrix} 9 & 0 \\ 6 & 9 \end{pmatrix} \end{pmatrix}
$$

In general, block matrices need not be diagonal matrices. For example,

$$
A = \left(\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ \hline a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{array}\right) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
$$

$$B = \left(\begin{array}{cc|cc|cc} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ \hline b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \end{array}\right) = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{pmatrix}$$

It is clear that

$$\begin{aligned} AB &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{pmatrix} \\ &= \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{pmatrix} \end{aligned}$$

where

$$C_{ij} = A_{i1}B_{ij} + A_{i2}B_{2j}, i = 1, 2; j = 1, 2, 3.$$

Partitioning matrices into block diagonal matrices, we can sometimes more easily compute the product of matrices.

Outer Product Expansions

Firstly, introduce two definitions:

Scalar Product or Inner Product: 1×1 matrix

$$x^T y = (x_1, x_2, \dots, x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + \cdots + x_n y_n$$

Outer Product:__________ matrix

$$xy^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} (y_1, y_2, \dots, y_n) = \underline{\hspace{3cm}}$$

Q1: If $X = \begin{pmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}, Y = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{pmatrix}$, please give XY^T.

New Way (Outer product Expansion:)

$$XY^T = \begin{pmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} (1, 2, 3) + \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} (2, 4, 1)$$

$$= \begin{pmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 4 & 1 \\ 8 & 16 & 4 \\ 4 & 8 & 2 \end{pmatrix}$$

Chapter 2

Determinants

2.1 The Determinant of A Matrix

Pre-Questions:

Q1: Can you solve the system of linear equations by your high school study?

$$\begin{cases} 2x + y = 7, \\ x - 3y = -2, \end{cases}$$

Q2: Can you solve the system of linear equations by your high school study?

$$\begin{cases} 2x - y + z = 0, \\ 3x + 2y - 5z = 1, \\ x + 3y - 2z = 4. \end{cases}$$

Q3: If there are four unknowns x, y, z and u in the corresponding system of linear equations, will you still take the same technique to do this question? Easy or tedious?

We will find proper technique for solving this kind of question.

1. Determinant of order 2.

Linear equation system in two unknowns x_1 and x_2:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2, \end{cases}$$

How to solve x_1 and x_2?

Your Technique: Eliminating x_2

Solutions:

$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$

$x_2 = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$

Denote $D = a_{11}a_{22} - a_{12}a_{21}$, where x_1,x_2 exist provided by $D \neq 0$.

New notations:

Determinant of order 2:

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

(1) Four elements are coefficients.
(2) Two rows and Two columns
(3) $a_{11}a_{22} \to$ principal diagonal
(4) $a_{12}a_{21} \to$ sub-diagonal
(5) D is a number

For example:

$$A = \begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix} = 1 \cdot 5 - (-2) \cdot 3 = 11$$

Similarly, we denote

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1a_{22} - b_2a_{12}$$

and

$$D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = b_2a_{11} - b_1a_{21}$$

So we have

$$x_1 = \frac{D_1}{D}$$

and

$$x_2 = \frac{D_2}{D}$$

Example 2.1. Please solve Q1 with new technique and notations.

2. Determinant of order 3.

Linear equation system in three unknowns x_1, x_2 and x_3:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

Simlar to analyze determinant of order 2, we have:

Determinant of order 3:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$-a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

For example:

$$D = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = 10$$

Certainly,

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

So,

$x_1 = \frac{D_1}{D}$, $x_2 = \frac{D_2}{D}$, and $x_3 = \frac{D_3}{D}$.

2.2 permutation

Definition 2.1 A **permutation** of the set of integers 1,2,...,n is an arrangement of these integers in some order without omissions repetitions.

Q:

Can you list the permutations of $\{1, 2, 3\}$? How many possibilities of them?

Example 2.2 Counting Inversions

Determine the number of inversions in the following permutations: (123), (231), (312), (321), (132), (213)

Inversion:
$\tau\mathbf{(123)} = 0 + 0 + 0 = 0$
$\tau\mathbf{(231)} = 0 + 0 + 2 = 2$
$\tau\mathbf{(312)} = 0 + 1 + 1 = 2$

$\tau(321) = 0 + 1 + 2 = 3$
$\tau(132) = 0 + 0 + 1 = 1$
$\tau(213) = 0 + 1 + 0 = 1$

What's the inversion of the permutaion (31542)?

Definition [redacted] A permutation is called **even** if the total number of inversion is an even integer and is called **odd** if the total number of inversions is an odd integer.

even permutation: **(123),(231),(312),**
odd permutation: (321),(132),(213).

[redacted] determinant with order n

we can rewrite

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum (-1)^{\tau} a_{1p_1} a_{2p_2} a_{3p_3}$$

where the first subscrips are arranged in the order of (1,2,3) and the second are in the order of a permutaion $(p_1 p_2 p_3)$ of $\{1, 2, 3\}$.

(a) six terms provided by $3! = 6$

(b) sign (+,-) in front of each term provided by $\tau(p_1p_2p_3)$
(c) 3^2 elements in determinant

Similarly,

We have

3. Determinant of order n:

Given n^2 numbers a_{ij}, $i, j = 1, \ldots, n.$ in n rows and n columns by

$$D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

$$= \sum (-1)^{\tau} a_{1p_1} a_{2p_2} a_{3p_3} \ldots a_{np_n}$$

Sometimes, we use $det(a_{ij})$ to indicate the determinant of order n, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

$(a)n!$ terms
(b) sign (+,-) in front of each term provided by

$\tau(p_1 p_2 \ldots p_n)$
(c) n^2 elements in determinant

3. Determinant of order n:

Given n^2 numbers a_{ij}, $i, j = 1, \ldots, n$. in n rows and n columns by

$$D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

Definition 2.2. Taking certain k definite rows and k definite columns of a determinant D of order n $(k < n)$, we construct a determinant of order k, called **a minor** of D of order k.

For example,
The minor of D of order 1,2 and 3

$$a_{12}, \qquad \begin{vmatrix} a_{12} & a_{14} \\ a_{32} & a_{34} \end{vmatrix}, \qquad \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

Definition 2.3. By omitting all the elements of the i-th row and j-th column at which the element a_{ij} of D locates, the resultant determinant M_{ij}, of order $n-1$, is called **the complement minor** of a_{ij} in D.

For example, the determinant of order 4

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Obviously, a_{11} and a_{23} are minor of order 1 and

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}, M_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

are minors of order 3.

We call M_{11} the complement minor of a_{11}, and M_{23} the complement minor of a_{23}.

Definition 2.4. $(-1)^{i+j}M_{ij}$ is called **the algebraic complement** or **(the algebraic) cofactor** of a_{ij} in D, denoted by A_{ij}:

$$A_{ij} = (-1)^{i+j}M_{ij}$$

For example, the determinant of order 4

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

the algebraic complements of a_{11}, a_{23} are respectively

$$A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$A_{23} = (-1)^{2+3} M_{23} = -M_{23} = -\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

Theorem 2.5. *The determinant of order n*

$$D = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

is equal to the sum of the products of all the elements of D in the i-th row and their algebraic complements:

$$D = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

It may also be developed about a column instead of a row: say the j-th column:

$$D = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

For example, developing A about its second column and third column we have, respectively,

$$A = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} + 3 \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= -4 + 18 - 4 = 10$$

and

$$A = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = 2 \begin{vmatrix} -4 & 3 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + 5 \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}$$

$$= -36 - 4 + 50 = 10$$

Corollary . *The sum of the products of all the elements in a certain row of a determinant of order $n(\geq 2)$ and the algebraic complements of the corresponding elements in another row is zero:*

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = 0 \qquad (i \neq j).$$

This is certainly true also for columns:

$$a_{1i}A_{1j} + a_{2i}A_{2j} + \cdots + a_{ni}A_{nj} = 0 \qquad (i \neq j).$$

2.3 Properties of Determinants

In this class we will use this determinant a lot of times which is

$$A = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = 30 + 2 - 24 - 12 + 20 - 6 = 10$$

1. **Property 1.**
When all the elements in a row (column) of a determinant are multiplied by a number k, the resultant determinant equals k times of the original one. Or, what is the same, if k is a common factor of all the elements in row (column) of a determinant, then k may be factored out of the determinant, i.e.,

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ ka_{i1} & \dots & ka_{in} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{i1} & \dots & a_{in} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

For example,

$$D = 2\begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 4 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 2\cdot 2 & 2\cdot 1 & 2\cdot 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix}$$

$= 20$

where $\begin{vmatrix} 4 & 2 & 4 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = 60-48+4-24+40-12 = 20$

or

$$D = 2\begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 4 \\ -4 & 3 & 2 \\ 2 & 3 & 10 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2\cdot 2 \\ -4 & 3 & 2\cdot 1 \\ 2 & 3 & 2\cdot 5 \end{vmatrix} = 20$$

where $\begin{vmatrix} 2 & 1 & 4 \\ -4 & 3 & 2 \\ 2 & 3 & 10 \end{vmatrix} = 60+4-48-24-12+40 =$ 20

2. **Property 2.**

If every element in the i-th row (column) of a determinant is written as the sum of two terms:

$$a_{ij} = b_j + c_j, \qquad j = 1, \ldots, n,$$

then the determinant is equal to the sum of two determinants in which the i-th rows (columns) are respectively $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$, the other rows(columns) keeping same.

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ b_1 + c_1 & b_2 + c_2 & \ldots & b_n + c_n \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ b_1 & b_2 & \ldots & b_n \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ c_1 & c_2 & \ldots & c_n \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

For example,

$$A = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 1+1 & 1+2 & 2+3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 5 + 5 = 10$$

where $\begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 12 - 8 + 1 - 6 + 8 - 2 = 5$

$\begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 18 + 1 - 16 - 6 - 4 + 12 = 5$

or

$$A = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 1+1 & 1 & 2 \\ -1-3 & 3 & 1 \\ 1+1 & 3 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 2 \\ -1 & 3 & 1 \\ 1 & 3 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 2 \\ -3 & 3 & 1 \\ 1 & 3 & 5 \end{vmatrix} = 6 + 4 = 10$$

where $\begin{vmatrix} 1 & 1 & 2 \\ -1 & 3 & 1 \\ 1 & 3 & 5 \end{vmatrix} = 15 + 1 - 6 - 6 - 3 + 5 = 6$

$$\begin{vmatrix} 1 & 1 & 2 \\ -3 & 3 & 1 \\ 1 & 3 & 5 \end{vmatrix} = 15 - 18 + 1 - 6 - 3 + 15 = 4$$

3. **Property 3.**
If any two rows (columns) of a determinant are interchanged, then only its sign is changed.
For example:

$$A = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = 10, \text{ and } D = \begin{vmatrix} 2 & 3 & 5 \\ -4 & 3 & 1 \\ 2 & 1 & 2 \end{vmatrix} = -10$$

where $\begin{vmatrix} 2 & 3 & 5 \\ -4 & 3 & 1 \\ 2 & 1 & 2 \end{vmatrix} = 12 + 6 - 20 - 30 + 24 - 2 =$ -10

or

$$A = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = 10, \text{ and } D = \begin{vmatrix} 1 & 2 & 2 \\ 3 & -4 & 1 \\ 3 & 2 & 5 \end{vmatrix} = -10$$

where $\begin{vmatrix} 1 & 2 & 2 \\ 3 & -4 & 1 \\ 3 & 2 & 5 \end{vmatrix} = -20+6+12+24-2-30 = -10$

By this conclusion we have

(1) If two rows of a determinant are identical then it is equal to zero.
For example,

$$A = \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} = 0$$

or

$$A = \begin{vmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 5 \end{vmatrix} = 0$$

(2) If the corresponding elements in rows of a determinant are proportional, then it equals zero.
For example,

$$A = \begin{vmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 1 & 3 & 5 \end{vmatrix} = 0$$

4. **Property 4.**
In a determinant, if we add k times of each element in a row to the corresponding element in another row, then the value of the determinant remains unchanged, i.e.,

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ a_{i1}+ka_{j1} & \dots & a_{in}+ka_{jn} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{i1} & \dots & a_{in} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \quad (j \neq i)$$

Properties 1-4 are valid for "columns"

in place of "row".

Example 2.6.

$$A = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ -4+2\cdot 2 & 3+1\cdot 2 & 1+2\cdot 2 \\ 2 & 3 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 2 \\ 0 & 5 & 5 \\ 2 & 3 & 5 \end{vmatrix} = 5\begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{vmatrix} = 10$$

Given a determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix},$$

if we change all its rows (columns) into columns (rows) with the orders preserved, then we get a new determinant

$$D' = \begin{vmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{vmatrix},$$

called the **transposed determinant** of D. Obviously, the transpose of D' is D. So D and D' are transposes of each other.

(1) a_{ji} is at i-th row and j-th column in D' which is at j-th row and i-column in D.
(2) The elements of D and D' on their principal diagonals remain unchanged.

5. **Property 5.**
The value of a determinant D is the same as its transpose D'.

For example,

$$A' = \begin{vmatrix} 2 & -4 & 2 \\ 1 & 3 & 3 \\ 2 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 5 & 2 \\ 2 & 5 & 3 \end{vmatrix} = 5 \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= 2 \times 5 = 10 = A$$

Q: We study 1-5 Properties for calculating determinant in simple way. How do we use these properties or we can say how do we calculate the value of any determinant in simple way?

Example 2.7.

$$A = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} \overset{r_2+2\times r_1}{=} \begin{vmatrix} 2 & 1 & 2 \\ 0 & 5 & 5 \\ 2 & 3 & 5 \end{vmatrix} \overset{r_3-r_1}{=} \begin{vmatrix} 2 & 1 & 2 \\ 0 & 5 & 5 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= 5 \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix} \overset{r_3 - 2\times r_2}{=} 5 \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 10$$

Given a determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix},$$

if we change all its rows (columns) into columns (rows) with the orders preserved, then we get a new determinant

$$D' = \begin{vmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{vmatrix},$$

called the **transposed determinant** of D. Obviously, the transpose of D' is D. So D and D' are transposes of each other.

(1) a_{ji} is at i-th row and j-th column in D' which is at j-th row and i-column in D.
(2) The elements of D and D' on their principal diagonals remain unchanged.

5. **Property 5.**
The value of a determinant D is the same as its transpose D'.

For example,

$$A' = \begin{vmatrix} 2 & -4 & 2 \\ 1 & 3 & 3 \\ 2 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 5 & 2 \\ 2 & 5 & 3 \end{vmatrix} = 5 \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$
$$= 2 \times 5 = 10 = A$$

Q: We study 1-5 Properties for calculating determinant in simple way. How do we use these properties or we can say how do we calculate the value of any determinant in simple way?

Example 2.8.

$$A = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} \overset{r_2+2\times r_1}{=} \begin{vmatrix} 2 & 1 & 2 \\ 0 & 5 & 5 \\ 2 & 3 & 5 \end{vmatrix} \overset{r_3-r_1}{=} \begin{vmatrix} 2 & 1 & 2 \\ 0 & 5 & 5 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= 5 \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix} \overset{r_3-2\times r_2}{=} 5 \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 10$$

Example 2.9. Calculate the determinant of order 4

$$\begin{vmatrix} 3 & 1 & -1 & 2 \\ -5 & 1 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 1 & -5 & 3 & -3 \end{vmatrix}$$

Solution:

After choosing property 4 to do action $c_1+2\times c_4$ and $c_4 + c_3$:

$$\begin{vmatrix} 7 & 1 & -1 & 1 \\ -13 & 1 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ -5 & -5 & 3 & 0 \end{vmatrix} = (-1)^{3+3} \begin{vmatrix} 7 & 1 & 1 \\ -13 & 1 & -1 \\ -5 & -5 & 0 \end{vmatrix}$$

Doing action $r_2 + r_1$:

$$D = \begin{vmatrix} 7 & 1 & 1 \\ -6 & 2 & 0 \\ -5 & -5 & 0 \end{vmatrix} = (-1)^{1+3} \begin{vmatrix} -6 & 2 \\ -5 & -5 \end{vmatrix}$$

Square Matrix and its Determinant

Suppose square matrix A of order n

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

then, we can get its corresponding determinant $|A|$ of order n, or $(detA)$.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Difference between Matrix of order n and Determinant of order n

We can calculate the value of Determinant, while the matrix is an array list composed of n^2 ordered numbers.
For example,

$$A = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{vmatrix} = 10.$$

and

$$A = \begin{pmatrix} 2 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 3 & 5 \end{pmatrix} \text{ is an array.}$$

Q2: If

$$A = \begin{pmatrix} -1 & 1 & 4 \\ 3 & -2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{pmatrix}$$

and $k = 2$

Please do these questions

a. $|A|$ and $|A^T|$?
b. $|kA|$ and $|A|$?
c. $|AB|$ and $|A| \cdot |B|$?

d. $|BA|$ and $|AB|$?

So we have several **properties**:

(1) $|A^T| = |A|$

(2) $|kA| = k^n|A|$

(3) $|AB| = |A| \cdot |B|$

(4) $|AB| = |BA|$

Example 2.10. If A is a square matrix of order 4 and $|A| = -2$, what is the value of $||A|A^2A^T|$?

Solution:

Obviously, $|A| = -2$ is a constant, so

$$\begin{aligned} ||A|A^2A^T| &= (-2)^4|A^2A^T| = (-2)^4|A^2||A^T| \\ &= (-2)^4|A||A||A| = (-2)^7 = -128 \end{aligned}$$

Definition 2.11. A is a square matrix of order n and $|A| \neq 0$, then we say A is nonsingular.

For example,

$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is nonsingular, and $B = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ is singular.

we NEXT discuss the existence of the inverse matrix.

Theorem 2.12. *The matrix A is invertible if and only if A is nonsingular.*

Example 1.

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Prove that A is invertible and find A^{-1}.

2.4 Cramer's Rule

Theorem 2.13. ***(Cramer's theorem)***
If the coefficient determinant $D \neq 0$, then it has the unique solution

$$x_1 = \frac{D_1}{D}, \cdots, x_n = \frac{D_n}{D},$$

where $D_i (i = 1, 2, \ldots, n)$ *is the determinant obtained when the i-th column* $a_{i1}, a_{i2}, \ldots, a_{in}$ *of* D *is replaced by the constants column* $b_1, b_2, \ldots, b_n$.

Example 2.14. Solve the system of linear equations

$$\begin{cases} 2x_1 + x_2 - 5x_3 + x_4 = 8 \\ x_1 - 3x_2 - 6x_4 = 9 \\ 2x_2 - x_3 + 2x_4 = -5 \\ x_1 + 4x_2 - 7x_3 + 6x_4 = 0 \end{cases}$$

Chapter 3

Vector Space

3.1 Rank of Matrix

Definition 3.1. A matrix is said to have *rank* r, if r is the largest order of nonzero minors of matrix A. The rank of a matrix A is denoted by $r(A)$. If the rank of a matrix of order n is n, then the matrix is called a *nonsingular matrix*, otherwise it is called a *singular matrix*.

For example:

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 4 & 6 & 0 \end{pmatrix}$$

A has a minor $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$ with order 2 and we have

$\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$

But we pick up any minor with order 3

for instance,

$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 6 \end{vmatrix} = 0.$

So we have rank of A, $r(A) = 2$.

When $A = O$, then $r(A) = 0$,

For any $m \times n$ matrix A we have

$$0 \leq r \leq min(m, n).$$

Theorem 3.2. *We do the elementary operations to a matrix, but it's rank remains unchanged.*

For example,

We have known

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 4 & 6 & 0 \end{pmatrix}$$

has rank 2.

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 4 & 6 & 0 \end{pmatrix} \overset{R_3+(-2)\times R_1}{\longrightarrow} B = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

then $r(A) = r(B) = 2$.

By this theorem, we will have the new technique, which is

$$A \overset{\text{elementary operations}}{\longrightarrow} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1r} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2r} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & a_{rr} & \dots & a_{rn} \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

then $r(A) = r$.

Example 3.3. Calculate $r(A)$, if

$$A = \begin{pmatrix} 1 & 3 & -1 & -2 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & -4 & 3 & 5 \end{pmatrix}$$

Solution:

$$A = \begin{pmatrix} 1 & 3 & -1 & -2 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & -4 & 3 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & -1 & -2 \\ 0 & -7 & 4 & 7 \\ 0 & -7 & 4 & 7 \\ 0 & -7 & 4 & 7 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 3 & -1 & -2 \\ 0 & -7 & 4 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So $r(A) = 2$.

Practice:

$$A = \begin{pmatrix} 1 & 0 & 1 & -1 & -2 \\ 2 & 1 & 3 & -1 & 6 \\ 1 & 1 & 2 & -2 & 5 \\ -1 & -1 & 1 & 0 & -1 \end{pmatrix}$$

3.2 Linearly Independence

Definition 3.4. A set of ordered numbers composed of n numbers a_1, a_2, $\ldots$, a_n, denoted by

$$\alpha = (a_1, a_2, \ldots, a_n),$$

is called an n-dimensional *vector*, where a_1, a_2, $\ldots$, a_n are called the *components* of α, a_i being called the i-th component.

In the example above

$\alpha = (1, 2, -1, 0), \beta = (2, -3, 1, 0), \gamma = (4, 1, -1, 0)$

Q: How to indicate $\alpha = (1, 2)$ in 2-space? and $\alpha = (1, 2, 3)$ in 3-space? Next to think about $\alpha = (1, 2, -1, 0)$ in space? What does it mean in analytical geometry?

1. A vector whose components are all zero is called a *zero vector*, usually denoted by o.
2. Two n-dimensional vectors $\alpha = (a_1, \ldots, a_n)$, $\beta = (b_1, \ldots, b_n)$ are said to be equal, if their corresponding components are equal. Thus we write it into $\alpha = \beta$, or

   $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ if $a_i = b_i$, $i = 1, 2, \ldots, n$.

For the operations of n-dimensional vectors
Definition 3.5. Let $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$

1. **Sum** of vector α and β: $\alpha + \beta = (a_1 + b_1, \ldots, a_n + b_n)$
2. **Difference** of vector α and β: $\alpha - \beta = (a_1 - b_1, \ldots, a_n - b_n)$
3. **scalar multiplication** of α and a number k: $k\alpha = \alpha k = (ka_1, \ldots, ka_n)$

Addition, subtraction and scalar multiplication of vectors are called *linear operations*.

Note: Only when the dimensions of two vectors are equal can we say these two vectors are equal or not equal and find their sum or difference.

For example:

(1,2,3)+(2,3,4,5) is meaningless.

we use the expression

$$(4, 1, -1, 0) = 2(1, 2, -1, 0) + (2, -3, 1, 0)$$

to show the relation among three equations.

or we rewrite it into

$$\gamma = 2\alpha + \beta$$

We also say that $2\alpha + \beta$ is a linear combination of vectors α and β, or γ is a linear expression of α and β. So we have

Definition 3.6. Let $\alpha_1, \ldots, \alpha_m$ be m vectors of dimension n, $k_1, \ldots, k_m$ be m constants. Then

$$k_1\alpha_1 + \cdots + k_m\alpha_m$$

is called a *linear combination* of vectors $\alpha_1, \ldots, \alpha_m$.

If $\alpha = k_1\alpha_1 + \cdots + k_m\alpha_m$ is called a *linear expression* of vectors $\alpha_1, \ldots, \alpha_m$.

For m vectors $\alpha_1, \ldots, \alpha_m$, if there is one vector is a linear combination of the $m-1$ remaining vectors (e.g. $\alpha_{m-1} = -\frac{k_1}{k_{m-1}}\alpha_1 - \cdots - \frac{k_m}{k_{m-1}}\alpha_m$), then we have

$$k_1\alpha_1 + \cdots + k_m\alpha_m = o, \tag{1}$$

where $k_1, \ldots, k_m$ are not all zero. Conversely, if the coefficients in (1) are not all zero, then among the m vectors $\alpha_1, \ldots, \alpha_m$ there at least exists a vector that is a linear combination of the $m-1$ remaining vectors. For example, if $k_m \neq 0$, then we obtain

$$\alpha_m = -\frac{k_1}{k_m}\alpha_1 - \cdots - \frac{k_{m-1}}{k_m}\alpha_{m-1}$$

i.e., α_m is a linear combination of $\alpha_1, \ldots, \alpha_{m-1}$.

Hence we have this **conclusion**:

There exists some vector among the m vectors $\alpha_1, \ldots, \alpha_m$ which is a linear combination of the $m-1$ remaining vectors if and only if the coefficients $k_1, \ldots, k_m$ which satisfy (1) are not all zero.

By this conclusion we have

Definition 3.7. Let α_1, ..., α_m be m vectors of dimension n. α_1, ..., α_m is said to be *linearly dependent*, if there exist constants k_1, ..., k_m not all zero, such that

$$k_1\alpha_1 + \cdots + k_m\alpha_m = o.$$

If no such constants k_1, ..., k_m exist, then α_1, ..., α_m are said to be *linearly independent.* That is to say, α_1, ..., α_m are linearly independent if (1) holds only when the constants k_1, ..., k_m are zero.

For example,

we have

$\alpha = (1, 2, -1, 0), \beta = (2, -3, 1, 0), \gamma = (4, 1, -1, 0)$

By

$$(4, 1, -1, 0) = 2(1, 2, -1, 0) + (2, -3, 1, 0)$$

i.e.

$$\gamma = 2\alpha + \beta \Rightarrow \gamma - 2\alpha - \beta = 0$$

Certainly, γ, α and β are linearly dependent.

Q: Please think about α and β? Linearly dependent or linearly independent?

Check:

$$k_1\alpha + k_2\beta = k_1(1, 2, -1, 0) + k_2(2, -3, 1, 0)$$

$$= (k_1 + 2k_2, 2k_1 - 3k_2, -k_1 + k_2, 0) = o,$$

$$k_1 + 2k_2 = 0, 2k_1 - 3k_2 = 0, -k_1 + k_2 = 0.$$

Certainly,

$$k_1 = 0, k_2 = 0$$

Hence α and β are linearly independent.

Example 3.8. Zero vector is a linear combination of any s vectors $\alpha_1, \cdots, \alpha_s$.

i.e.

$$o = 0 \cdot \alpha_1 + 0 \cdot \alpha_2 + \cdots + 0 \cdot \alpha_s.$$

Example 3.9. In vectors α_1, α_2, $\ldots$, α_s, any vector α_j $(1 \leq j \leq s)$ is a linear combination of other remaining vectors.

i.e.

$$\alpha_j = 0 \cdot \alpha_1 + \cdots + 1 \cdot \alpha_j + \cdots + 0 \cdot \alpha_s.$$

Suppose (m,n)-matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

then we call any row $(a_{i1}, a_{i2}, \cdots, a_{in})$ a row vector in n dimensions. Certainly, we call any column $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$ a column vector in m dimensions. Sometimes, for convenience, column vectors are also written horizontally.

i.e., $(a_{1j}, a_{2j}, \cdots, a_{mj})^T$

Review: Definition of **Rank** of matrix

If the rank of a matrix of order n is n, then the matrix is called a **nonsingular** matrix, otherwise it is called a **singular** matrix.

Theorem 3.10. *The m row vectors of an (m,n)-matrix A are linearly dependent if and* *only if the rank of A is less than m.*

In this theorem, we replace the rows by columns, it is still true, i.e., n-columns of an (m, n)-matrix A are linearly dependent if and only if the rank of A is less than n.

Example 1,

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 3 \end{pmatrix}$$

So $r(A) = 2$, and there are two row vectors. By theorem, we say they are linearly independent, but its three column vectors are linearly dependent because of $2 < 3$.

Example 2,

$$B = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ 4 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ 4 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 3 \\ 0 & -7 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

So $r(B) = 2$.

Then its three row vectors are linearly dependent, so are the three columns.

In Example 2, $r(B) = 2$ and order of B is 3, so we have $|B| = 0$, That is easy to see

1. There exist n n-dimensional row vectors $\alpha_{1j}, \cdots, \alpha_{nj}$ $(j = 1, \cdots, n)$ which are linearly dependent if and only if

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = 0$$

2. There exist n n-dimensional row vectors $\alpha_{1j}, \cdots, \alpha_{nj}$ $(j = 1, \cdots, n)$ which are linearly independent if and only if

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

3. There are n vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$, and each vector α_j is in m-dimensional, i.e.

$$\alpha_j = (a_{1j}, a_{2j}, \cdots, a_{mj})^T (j = 1, 2, \cdots, n)$$

If $m < n$, then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly dependent. (By theorem 3.10)

Q:(1) Try to check vectors $\alpha_1 = (1, 2, -1, 5)$, $\alpha_2 = (2, -1, 1, 1)$, $\alpha_3 = (4, 3, -1, 11)$ linearly dependent?

(2) $\alpha_1 = (1, 2, 0, 1)$, $\alpha_2 = (1, 3, 0, -1)$, $\alpha_3 = (-1, -1, 1, 0)$ linearly dependent?

Theorem 3.11. *The rank of a matrix A is equal to r if and only if there are r row (column) vectors in A which are linearly independent, and any $r + 1$ row (column) vectors (if any) are all linearly dependent.*

For example:

In Example 1:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 3 \end{pmatrix}$$

So $r(A) = 2$, by theorem, we say (1,2,-1) and

(2,-3,1) are linearly independent.

But in

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ 4 & -6 & 2 \end{pmatrix}$$

We have

$$B = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ 4 & -6 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 3 \\ 0 & -14 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Certainly, $r(B) < 3$, then (1,2,-1), (2,-3,1) and more(4,-6,2) are linearly dependent.

3.3 Some definitions and examples

Definition 3.12. Let V be a set on which the operations of addition and scalar multiplication are defined. By this we mean that, with each pair of elements x and y in V, we can associate a unique element $x+y$ that is also in V, and with each element x in V and each scalar α, we can associate a unique element αx in V. The set V together with the operations of addition and scalar multiplication is said to form a **vector space** if the following axioms are satisfied.

A1. $x + y = y + x$ for any x and y in V.

A2. $(x + y) + z = x + (y + z)$ for any x, y, z in V.

A3. There exists an element 0 and V such that $x + 0 = x$ for each $x \in V$.

A4. For each $x \in V$, there exists an element $-x$ in V such that $x + (-x) = 0$.

A5. $\alpha(x + y) = \alpha x + \alpha y$ for each scalar α and any x and y in V.

A6. $(\alpha + \beta)x = \alpha x + \beta x$ for any scalar α and β and any $x \in V$.

A7. $(\alpha\beta)x = \alpha(\beta x)$ for any scalars α and β and any $x \in V$

A8. $1 \cdot x = x$ for all $x \in V$.

Definition 3.13. If S is a nonempty subset of a vector space V, and S satisfies the following conditions:

(i) $\alpha x \in S$ whenever $x \in S$ for any scalar α

(ii) $x + y \in S$ whenever $x \in S$ and $y \in S$.

then S is said to be a **subspace** of V.

Example 3.14. Let $S = \{(x_1, x_2, x_3)^T | x_1 = x_2\}$. S is nonempty since $x = (1, 1, 0)^T \in S$. To show that S is a subspace of R^3, we need to verify that the two closure properties hold

(i) If $x = (a, a, b)^T$ is any vector in S, then

$$\alpha x = (\alpha a, \alpha a, \alpha b)^T \in S$$

(ii) If $(a, a, b)^T$ and $(c, c, d)^T$ are arbitrary elements of S, then

$$(a, a, b)^T + (c, c, d)^T = (a+c, a+c, b+d)^T \in S$$

Since S is nonempty and satisfies the two closure conditions, it follows that S is a subspace of R^3.

Definition 3.15. Let A be an $m \times n$ matrix. Let $N(A)$ denote the set of all solution to the homogeneous system $AX = 0$. Thus

$$N(A) = \{x \in R^n | Ax = 0\}$$

We claim that $N(A)$ is a subspace of R^n. Clearly, $0 \in N(A)$, so $N(A)$ is nonempty. If $x \in N(A)$ and α is a scalar, then

$$A(\alpha x) = \alpha Ax = \alpha 0 = 0$$

and hence $\alpha x \in N(A)$. If x and y are elements of $N(A)$, then

$$A(x + y) = Ax + Ay = 0 + 0$$

Therefore, $x + y \in N(A)$. It follows then that $N(A)$ is a subspace of R^n. The set of all solutions to the homogeneous system $Ax = 0$ forms a subspace of R^n. The subspace $N(A)$ is called the *nullspace* of A.

Example 3.16. Determine $N(A)$ if

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

Solution

Using Gauss-Jordan reduction to solve $Ax = 0$, we obtain

$$\begin{pmatrix} 1 & 1 & 1 & 0 & \vdots 0 \\ 2 & 1 & 0 & 1 & \vdots 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 & \vdots 0 \\ 0 & -1 & -2 & 1 & \vdots 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1 & 1 & \vdots 0 \\ 0 & -1 & -2 & 1 & \vdots 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 1 & \vdots 0 \\ 0 & 1 & 2 & -1 & \vdots 0 \end{pmatrix}$$

We set $x_3 = \alpha$, $x_4 = \beta$

Then $x = \begin{pmatrix} \alpha - \beta \\ -2\alpha + \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

is a solution to $Ax = 0$. The vector space $N(A)$ consists of all vectors of the form

$$\alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

where α and β are scalars.

Definition 3.17. Let $v_1, \ldots, v_n$ be n vectors in V, $\alpha_1, \ldots, \alpha_n$ be n scalars. Then

$$\alpha_1 v_1 + \cdots + \alpha_n v_n$$

is called a *linear combination* of vectors $v_1, \ldots, v_n$.

The set of all linear combinations of $v_1, \ldots, v_n$ is called the **span** of $v_1, \ldots, v_n$. The span of $v_1, \ldots, v_n$ will be denoted by $\mathbf{Span}(v_1, \ldots, v_n)$.

In example,

The nullspace of A was the span of the vector

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Definition 3.18. The vectors $v_1, \ldots, v_n$ form a **basis** for a vector space V if and only if

(i) $v_1, \ldots, v_n$ are linearly independent.

(ii) $v_1, \ldots, v_n$ span V.

Example 3.19. The *standard basis* for R^3 is $\{e_1, e_2, e_3\}$; however, there are many bases that we could choose for R^3. For example,

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Chapter 4

Eigenvalues

4.1 Eigenvalues and Eigenvectors

Q1:

$$A = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$$

Please give the expression of $|\lambda E - A|$?

Q2:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Please give the expression of $|\lambda E - A|$?

Q3:

If we suppose $|\lambda E - A| = 0$, can you calculate λ?

Definition 4.1. $f(\lambda)$ is a polynomial of degree n in λ, it is said to be the *characteristic polynomial* of the matrix A. The equation $f(\lambda) = 0$ is called the *characteristic* equation of A. The solution(roots) of characteristic equation are known as the *eigenvalues(characteristic roots)* of A, the k-multiple roots of $f(\lambda) = 0$ are called the *eigenvalues (characteristic roots)* of *multiplicity* k of A.

Example 4.2. Find eigenvalues of the scalar matrix

$$A = \begin{bmatrix} a & & \\ & a & \\ & & a \end{bmatrix}.$$

Solution:

$$|\lambda E - A| = \begin{vmatrix} \lambda - a & & \\ & \lambda - a & \\ & & \lambda - a \end{vmatrix} = (\lambda - a)^3,$$

$\lambda = a$ is an eigenvalue of *multiplicity 3 (3-multiple eigenvalue)*.

In general, the eigenvalues of any scalar matrix of order n are eigenvalues of multiplicity n.

Definition 4.3. Assume λ_0 is an eigenvalue of A. Then

$$f(\lambda_0) = |\lambda_0 E - A| = 0.$$

Hence the homogeneous system of linear equations

$$(\lambda_0 E - A)X = 0, \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

i.e.

$$\begin{cases} (\lambda_0 - a_{11})x_1 - \cdots - a_{1n}x_n = 0, \\ \hfill \cdots\cdots \\ a_{n1}x_1 - \cdots + (\lambda_0 - a_{nn})x_n = 0, \end{cases}$$

has nonzero solution vectors. Any one of its nonzero solution vectors is called an *eigenvector* of A associated with the eigenvalue λ_0.

For example,

Please calculate the *eigenvector* in **Q2.**

Solution:

By

$$|\lambda E - A| = \begin{vmatrix} \lambda+1 & -1 & 0 \\ 4 & \lambda-3 & 0 \\ -1 & 0 & \lambda-2 \end{vmatrix} = 0$$

We have $(\lambda-2)(\lambda-1)^2 = 0$, so $\lambda_1 = 2$, and $\lambda_2 = \lambda_3 = 1$.

When $\lambda_1 = 2$,

We solve $(2E - A)X = 0$,

$$2E-A = \begin{bmatrix} 3 & -1 & 0 \\ 4 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We set $x_3 = \alpha$. then $x = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Certainly, we can say

$X_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector belongs to $\lambda_1 = 2$.

So all eigenvectors associated with $\lambda_1 = 2$ are

$$\alpha X_1 = \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where $\alpha \neq 0$.

Q: Can you indicate $N(2E - A)$?

When $\lambda_2 = \lambda_3 = 1$, we solve $(E - A)X = 0$

$E - A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -4 \\ 0 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

We set $x_3 = -\beta$, and $x = \begin{bmatrix} \beta \\ 2\beta \\ -\beta \end{bmatrix}$

We have $X_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an eigenvector belongs to $\lambda_2 = \lambda_3 = 1$.

So all the eigenvectors

$$\beta X_2 = \beta \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

where $\beta \neq 0$.

Q: Can you indicate $N(E - A)$?

we indicate eigenspace corresponding to λ by $N(\lambda I - A)$.

Practice:

Find the eigenvalues and corresponding eigenspaces.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

Solution:

$f(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & -1 & 1 \\ 2 & \lambda - 4 & 2 \\ 2 & -2 & \lambda \end{vmatrix} = (\lambda - 2)^2(\lambda - 1)$

So $\lambda_1 = \lambda_2 = 2$, and $\lambda_3 = 1$

When $\lambda_1 = \lambda_2 = 2$,

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 2 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

When $\lambda_3 = 1$,

$$\begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 2 \\ 2 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $X_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Are X_1, X_2, X_3 linearly independent or not?

Several conclusions:

Let A be an $n \times n$ matrix and λ be a scalar. The following statements are equivalent:

(a) λ is an eigenvalue of A.

(b) $(A - \lambda I)x = 0$ has a nontrivial solution.

(c) $N(A - \lambda I) \neq 0$

(d) $A - \lambda I$ is singular.

(b) $\det(A - \lambda I) = 0$

Example 4.4. Matrix A with order n is singular if and only if A has a eigenvalue 0.

Example 4.5. Suppose A is an idempotent matrix, i.e. $A = A^2$. Prove that the eigenvalues of A equals either zero or one.

Proof.

Suppose λ is an eigenvalue of A, X is an eigenvector of A associated with λ, i.e.

$$AX = \lambda X$$

Then $AX = A^2X = \lambda AX = \lambda^2 X$. Hence $\lambda X = \lambda^2 X$ or $(\lambda^2 - \lambda)X = 0$. As $X \neq 0$, $\lambda^2 - \lambda = \lambda(\lambda - 1) = 0$. Hence $\lambda = 1$ or $\lambda = 0$.

Practice: Page 311. 4

Several Conclusions:

1.

Theorem 4.6. *Suppose matrix A with order n, its eigenvectors X_1, X_2, $\cdots$, X_m associated with distinct eigenvalues λ_1, λ_2, $\cdots$, λ_m are linearly independent.*

2.

Theorem 4.7. *If A with order n is similar to diagonal matrix B if and only if for each k-multiple eigenvalues λ_i, the rank of matrix $\lambda_i I - A$ is n-k.*

4.2 Diagonalization of Matrices

Definition 4.8. We say B is *similar* to A, denoted by $A \sim B$, if there is an invertible matrix P such that

$$B = P^{-1}AP.$$

From $B = P^{-1}AP$, we have $P^{-1}A = BP^{-1} = Q$. Hence we have

$$A = PQ, \qquad B = QP$$

where P is nonsingular.

For example,

$$A = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 \\ 1 & -5 \end{bmatrix}$$

then

$$P^{-1} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} = B$$

So $A \sim B$

Theorem 4.9. *If $A \sim B$ with order n, then A and B have same eigenvalues.*

Proof.

$A \sim B$, there exists a nonsingular P, then $P^{-1}AP = B$.

$$|\lambda I - B| = |\lambda I - P^{-1}AP| = |P^{-1}(\lambda I)P - P^{-1}AP|$$

$$= |P^{-1}(\lambda I - A)P| = |P^{-1}||\lambda I - A||P|$$
$$= |\lambda I - A|$$

Suppose $A = (a_{ij})$ is a matrix of order n. If it is similar to a diagonal matrix, we can write

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

or

$$AP = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

Writing the n column vectors of P as $X_1, \cdots,$ X_n in their proper order, i.e., $P = (X_1, X_2, \cdots, X_n)$.

$$AP = (AX_1, \cdots, AX_n),$$

$$P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = (\lambda_1 X_1, \cdots, \lambda_n X_n).$$

Hence we have

$$(AX_1, \cdots, AX_n) = (\lambda_1 X_1, \cdots, \lambda_n X_n),$$

and so

$$AX_i = \lambda_i X_i \text{ or } (\lambda_i E - A)X_i = 0, \ i = 1, \cdots, n$$

Therefore the column vectors X_i of matrix P are all eigenvectors of A.

Since P is nonsingular, $X_1, X_2, \cdots, X_n$ are n linearly independent eigenvectors of A. That is to say, if the matrix A of order n is similar to diagonal matrix, then it has n linearly independent eigenvectors.

Conversely, if A has n linearly independent eigenvectors $X_1, \cdots, X_n$, then $AX_i = \lambda_i X_i$ ($i =$

$1, 2, \ldots, n$) Let $P = (X_1, \cdots, X_n)$. Obviously P is nonsingular matrix. As

$$AP = (AX_1 \cdots AX_n) = (\lambda_1 X_1 \cdots \lambda_n X_n)$$

$$= (X_1, \cdots, X_n) \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

A is similar to the diagonal matrix $P^{-1}AP$. Thus we have the following basic theorem

Theorem 4.10. *A matrix A of order n is similar to a diagonal matrix if and only if A has n linearly independent eigenvectors. If A has n linearly independent eigenvectors $X_1, X_2, \ldots, X_n$, $AX_i = \lambda_i X_i$, $P = (X_1, \cdots, X_n)$, then*

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

If the order of $\lambda_1, \cdots, \lambda_n$ is changed, then the order of $X_1, \ldots, X_n$ is changed accordingly.

In previous example,

$$A = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$$

has two distinct eigenvalues $\lambda_1 = 4$, $\lambda_2 = -2$ corresponding eigenvectors $\alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\alpha_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$.

If we take

$B_1 = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$, then $P = (\alpha_1, \alpha_2) = \begin{bmatrix} 1 & 1 \\ 1 & -5 \end{bmatrix}$, we have $P^{-1}AP = B_1$.

If we take $B_2 = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$, then $P = (\alpha_2, \alpha_1) = \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}$.

Example 4.11. Find a diagonal matrix similar to

$$\begin{bmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{bmatrix}$$

and the matrix P used. And find A^{100}.

4.3 Orthogonal Diagonalization

Orthogonal Matrix

A real square matrix is called an **orthogonal matrix** if it satisfies $AA' = A'A = E$.

For example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

It is to see [illegible]he definition that if A is orthogonal, then

$$|AA'| = |A||A'| = |A|^2 = 1$$

So

$$|A| = 1 \quad \text{or} \quad |A'| = -1$$

or -1 or -1

Q1: If A is orthogonal, what about A'?

Q2: If A and B are both orthogonal, what about AB?

Q3: If A is orthogonal, what about $-A$?

Q4: Please to check:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$$

We have several conclusions:

(1) There is an orthogonal matrix, so is it's transpose.

(2) The product of orthogonal matrices is also an orthogonal matrix.

(3) There is an orthogonal matrix, so is it's negative.

(4) If we interchange any two rows or two columns in an orthogonal matrix, we still get another orthogonal one either.

Q:

Find the eigenvectors of symmetric matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

and to check $X_i^T X_j$, where $i = 1, 2, 3$, $j = 1, 2, 3$ and $i \neq j$.

Definition 4.12. If $\alpha^T \beta = 0$, then we say α and β are orthogonal.

Check $Au \cdot v = u \cdot A^T v$ where A is $n \times n$ matrix, u and v are $n \times 1$ column vectors.

For example,

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}, u = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, v = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

Then,

$$Au = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$$

$$A^T v = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$$

We obtain

$$Au \cdot v = 11, u \cdot A^T v = 11$$

Theorem 4.13. *The eigenvectors of a real symmetric matrix associated with distinct eigenvalues are orthogonal eigenvectors.*

Proof: Let v_1 and v_2 be eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2 of the matrix A. We want to show $v_1 \cdot v_2 = 0$.

$$Av_1 \cdot v_2 = v_1 \cdot A^T v_2 = v_1 \cdot Av_2$$

Certainly,

$$\lambda_1 v_1 \cdot v_2 = v_1 \cdot \lambda_2 v_2$$

Simplify it,

$$(\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0$$

But $\lambda_1 - \lambda_2 \neq 0$, since λ_1 and λ_2 were assumed distinct. It follows $v_1 \cdot v_2 = 0$.

Theorem 4.14. *Suppose that a matrix A is real symmetric matrix, then there exists an orthogonal matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.*

Example 4.15. Find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

Solution:

$$f(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & 2 & 0 \\ 2 & \lambda - 2 & 2 \\ 0 & 2 & \lambda - 3 \end{vmatrix} = 0$$

We have $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 5$.

When $\lambda_1 = -1$, $(-E - A)X = 0$, we have $\alpha_1 = (2, 2, 1)^T$.

When $\lambda_2 = 2$, $(2E - A)X = 0$, we have $\alpha_2 = (2, -1, -2)^T$.

When $\lambda_3 = 5$, $(5E - A)X = 0$, we have $\alpha_3 = (1, -2, 2)^T$.

It very easy to check α_1, α_2 and α_3 are orthogonal vectors.

Orthogonalize and normalize:

Now to normalize we get

$\beta_1 = \frac{1}{||\alpha_1||}\alpha_1 = (2/3, 2/3, 1/3)^T$,

$\beta_2 = \frac{1}{||\alpha_2||}\alpha_2 = (2/3, -1/3, -2/3)^T$,

$\beta_3 = \frac{1}{||\alpha_3||}\alpha_2 = (1/3, -2/3, 2/3)^T$.

Let

$$Q = (\beta_1, \beta_2, \beta_3) = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$

then $Q^{-1}AQ = Q^TAQ = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

We know in **Example 4.13** $\alpha_1 = (2, 2, 1)^T$, $\alpha_2 = (2, -1, -2)^T$ and $\alpha_3 = (1, -2, 2)^T$ are orthogonal vectors, are they linearly dependent or linearly independent?

Theorem 4.16. *Suppose $\alpha_1, \ldots, \alpha_n$ are n nonzero pairwise orthogonal vectors. Then $\alpha_1, \ldots, \alpha_n$ are linearly independent.*

Can you find an orthogonal matrix to diagonalize the matrix?

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

For example,

$\alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\alpha_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are not orthogonal, can you use linear combinations of vector $\alpha_1, \alpha_2, \alpha_3$ to find 3 pairwise orthogonal unit vectors?

1. We can take $\beta_1 = \alpha_1$, and we want to find

β_2 which is orthogonal to β_1 in the linear combination $\beta_1(\alpha_1)$ and α_2 (β_2 is indicated by α_1 and α_2).

We can suppose

$$\beta_2 = k\beta_1 + \alpha_2 \tag{1}$$

then

$$\beta_1^T \beta_2 = k\beta_1^T \beta_1 + \beta_1^T \alpha_2 = 0.$$

Since $\beta_1 \neq 0$, we get

$$k = -\frac{\beta_1^T \alpha_2}{\beta_1^T \beta_1}$$

(Substituting it in the right hand side of (1) and you can check β_1 and β_2 are orthogonal.)

2. We look for β_3 in the linear combinations of $\beta_1, \beta_2, \alpha_3$.

   Suppose

$$\beta_3 = k_1\beta_1 + k_2\beta_2 + \alpha_3 \tag{2}$$

From

$$\beta_1^T\beta_3 = k_1\beta_1^T\beta_1 + k_2\beta_1^T\beta_2 + \beta_1^T\alpha_3 = 0,$$

$$\beta_2^T\beta_3 = k_1\beta_2^T\beta_1 + k_2\beta_2^T\beta_2 + \beta_2^T\alpha_3 = 0,$$

We get

$$k_1 = -\frac{\beta_1^T\alpha_3}{\beta_1^T\beta_1}, k_2 = -\frac{\beta_2^T\alpha_3}{\beta_2^T\beta_2}.$$

(Substituting it in the right hand side of (2) and you can check β_3 is orthogonal to β_1 and β_2.)

The process of finding the nonzero pairwise orthogonal vectors $\beta_1, \beta_2, \beta_3$ from the linear combinations of linear independent vectors $\alpha_1, \alpha_2, \alpha_3$ is known as *orthogonalizing* vectors $\alpha_1, \alpha_2, \alpha_3$.

Please find $\beta_1, \beta_2, \beta_3$ in example.

Solution:

$$\beta_1^T\beta_1 = (1, 1, 0, 0)\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 2$$

$$\beta_1^T\alpha_2 = (1,1,0,0)\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = 1.$$

We have

$$k = k = -\frac{\beta_1^T\alpha_2}{\beta_1^T\beta_1} = -\frac{1}{2},$$

and hence

$$\beta_2 = -\frac{1}{2}\beta_1 + \alpha_2 = \begin{bmatrix}1/2\\-1/2\\1\\0\end{bmatrix}.$$

then let $\beta_3 = k_1\beta_1 + k_2\beta_2 + \alpha_3$. As

$$k_1 = -\frac{\beta_1^T\alpha_3}{\beta_1^T\beta_1} = \frac{1}{2}, k_2 = -\frac{\beta_2^T\alpha_3}{\beta_2^T\beta_2} = \frac{1}{3}.$$

We have

$$\beta_3 = \frac{1}{2}\beta_1 + \frac{1}{3}\beta_2 + \alpha_3 = \begin{bmatrix} -1/3 \\ 1/3 \\ 1/3 \\ 1 \end{bmatrix}.$$

Next we normalize $\beta_1, \beta_2, \beta_3$, obtaining the pairwise orthogonal unit vectors

$$\frac{1}{\sqrt{1^2+1^2}}\beta_1, \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}\beta_2, \frac{1}{\sqrt{\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+1}}\beta_3$$

i.e.,

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2\sqrt{3} \\ 1/2\sqrt{3} \\ 1/2\sqrt{3} \\ \sqrt{3}/2 \end{bmatrix}$$

Example 4.17. Using orthogonal matrix reduce

$$A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

into a diagonal matrix.

Solution:

Since $|\lambda E - A| = (\lambda - 1)^3(\lambda + 3)$, A has eigenvalues $\lambda_1 = 3$, and $\lambda_2 = \lambda_3 = \lambda_4 = 1$.

Solveing $(-3E - A)X = 0$, we obtain the eigenvectors $\alpha_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ which is orthogonal to other 3 eigenvectors $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\alpha_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

We get $\beta_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$ after normalizing it.

Orthogonalizing and normalizing, in previous example, we have

$$\beta_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \beta_3 = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \end{bmatrix}, \beta_4 = \begin{bmatrix} -1/2\sqrt{3} \\ -1/2\sqrt{3} \\ 1/2\sqrt{3} \\ \sqrt{3}/2 \end{bmatrix}$$

Thus A is similar to

$$Q^T A Q = \begin{bmatrix} -3 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

where

$$Q = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/\sqrt{6} & -1/2\sqrt{3} \\ -1/2 & 1/\sqrt{2} & -1/\sqrt{6} & 1/2\sqrt{3} \\ -1/2 & 0 & 2/\sqrt{6} & 1/2\sqrt{3} \\ 1/2 & 0 & 0 & \sqrt{3}/2 \end{bmatrix}$$

4.4 Quadratic Forms

Q: Please do the the product

$$(x_1, x_2, x_3) \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & -\frac{3}{2} \\ \frac{1}{2} & -\frac{3}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Certainly, we can give $f(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + 2x_2^2 - 3x_2x_3$ new name which is *quadratic form* in two variables.

In **Q**, suppose $\mathbf{x}^T = (x_1, x_2, x_3)$, $A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & -\frac{3}{2} \\ \frac{1}{2} & -\frac{3}{2} & 0 \end{bmatrix}$, then we have

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$$

Obviously, A is a symmetric matrix.

In general, a quadratic form in n variables can be written as

$$f(x_1, \ldots, x_n) = a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n$$
$$+a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n$$
$$+\cdots\cdots$$
$$+a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2$$

If we suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, by the product of matrices

$$\mathbf{x}^T A\mathbf{x} = (x_1, x_2, \ldots, x_n) \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n$$
$$+a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n$$
$$+\cdots\cdots$$
$$+a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2$$

If $a_{ij} = a_{ji}$, $(i, j = 1, 2, 3, \cdots, n)$(or $A^T = A$), then

$$\begin{aligned}\mathbf{x}^T A\mathbf{x} = a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\ +a_{22}x_2^2 + \cdots \qquad\qquad + 2a_{2n}x_2x_n \\ + \cdots\cdots \qquad\qquad \\ +a_{nn}x_n^2\end{aligned}$$

If we have $f(x, y, z) = x^2 + 2y^2 + 5z^2 + 2xy + 6yz + 2zx$, then it is *quadratic form* in three variables.

Can you write it into the form $f(x, y, z) = (x, y, z)A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, what's A here?

Conclusion 1:

Quadratic Form → Symmetric Matrix

Practice:

Please give the matrix from the following quadratic form:

(1)$x_1^2 - 2x_1x_2 + 3x_1x_3 - 2x_2^2 + 8x_2x_3 + 3x_3^2$
(2)$x_1x_2 - x_1x_3 + 2x_2x_3 + x_4^2$

Conclusion 2:

Symmetric Matrix $\to$ Quadratic Form

Practice:

Please give the quadratic form the following matrix from:

$$A = \begin{bmatrix} 1 & -1 & -3 & 1 \\ -1 & 0 & -2 & \frac{1}{2} \\ -3 & -2 & \frac{1}{3} & -\frac{3}{2} \\ 1 & \frac{1}{2} & -\frac{3}{2} & 0 \end{bmatrix}$$

Exercise:

Write the following quadratic form into $X^T AX$.

(1)$2x^2 + 3y^2 + z^2 + xy - 2xz + 3yz$

(2)$x_1^2 + 2x_2^2 + x_3^2 + x_1x_2 - 2x_1x_3 + 3x_2x_3$

Definition 4.18. When a real quadratic form $f = X'AX$ is positive definite, negative definite, or indefinite, the real symmetric matrix A is called a *positive definite matrix*, a *negative definite matrix*, or an *indefinite matrix* respectively.

Theorem 4.19. *A real quadratic form*

$$f(x_1, \cdots, x_n) = X'AX$$

is positive definite, if and only if the principal minors of the matrix $A = (a_{ij})$ *in the upper left corner are all positive, that is,*

$$A_1 = a_{11} > 0, A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0,$$

$$A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \cdots, A_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0.$$

For example,

Matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

is a positive definite matrix.

Because

$$A_1 = 1 > 0, A_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 > 0, A_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = 1 > 0$$

Theorem 4.20. *f is negative definite if and only if all the principal minors of its matrix A whose orders are odd numbers are less than zero and all the principal minors of A whose orders are even numbers are greater than zero.*

$$A_1 = a_{11} < 0, A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0,$$

$$A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0, \cdots, A_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

(>0 for n is even) (< 0 for n is odd).

For example,

Matrix

$$A = \begin{bmatrix} -5 & 2 & 2 \\ 2 & -6 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

is a negative matrix because

$$A_1 = -5, A_2 = 26, A_3 = -80$$

For matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 1 & -3 & 0 \end{bmatrix}$$

It is indefinite matrix as $a_{11} = 0$.

Exercise:

Please go back to determine the quadratic forms on page 135 and 136 positive definite, negative definite or indefinite?